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On the real theory of four-dimensional conformal structures

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Abstract

The conformal structures $CO(4,0), CO(1,3)$ and $CO(2,2)$ are studied on a real manifold M, $\dim M = 4$. On M isotropic fiber bundles E_α and E_β are constructed. These bundles are real for the $CO(2,2)$-structure, and they satisfy the condition $\overline{E}_\alpha = E_\beta$ for the $CO(1,3)$-structure, and the conditions $\overline{E}_\alpha = E_\alpha$, $\overline{E}_\beta = E_\beta$ for the $CO(4)$-structure. The tensor C of conformal curvature splits into two subtensors C_α and C_β which are the curvature tensors of the bundles E_α and E_β, respectively. These subtensors satisfy the same conditions as the bundles E_α and E_β. Conformally semiflat and flat structures and their geometrical characteristics are studied. The principal 2-directions are defined, and conditions for their integrability are obtained. These investigations for the $CO(1,3)$-structure are connected with Petrov's classification of Einstein's spaces.

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0. Introduction

0.1

Four-dimensional conformal structures play an important role in general relativity. Space-time in general relativity is a four-dimensional Riemannian manifold of signature $(1,3)$. Since many features of general relativity are of a conformal invariant nature, it is interesting to study pseudoconformal structures of signature $(1,3)$. Along with these kinds of conformal structures, on a real four-dimensional conformal structure, one also can consider conformal structures of signatures $(4,0)$ and $(2,2)$. By means of complexification of a manifold M,

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all these structures can be reduced to one of them, for example, to the structure of signature $(4, 0)$ or $(2, 2)$. This was done in many investigations (see, for example, [AHS 78,Gi 83, P 77]).

Unlike the previous investigations, in the present paper, we consider four-dimensional conformal structures on a real manifold M and study their common properties and the differences between them. Moreover, we also apply complexification not of the manifold M itself but only of its tangent spaces $T_x(M)$, and consider in these spaces coordinate transformations preserving the real part of these tangent spaces.

In the paper, we study conformal structures of signatures $CO(4, 0) = CO(4)$, $CO(1, 3)$ and $CO(2, 2)$ in a parallel way, consider their isotropic cones and construct their isotropic fiber bundles E_α and E_β. These bundles are real for the $CO(2, 2)$-structure, and they are complex for the $CO(1, 3)$- and $CO(4)$-structures. Moreover, for the $CO(1, 3)$-structure, these bundles satisfy the condition $\overline{E}_\alpha = E_\beta$, and for the $CO(4)$-structure, the conditions $\overline{E}_\alpha = E_\alpha$ and $\overline{E}_\beta = E_\beta$. All these structures are G-structures of first order with seven-parameter structure groups.

Next, we deduce the structure equations for the $CO(2, 2)$-structure and compute the components of its tensor of conformal curvature. This tensor splits into two real subtensors C_α and C_β which are the curvature tensors of the isotropic fiber bundles E_α and E_β.

Complexifying appropriately the structure equations of the $CO(2, 2)$-structure, we also find the forms and the tensor of conformal curvature of $CO(1, 3)$- and $CO(4)$-structures. However, these forms and the tensor are complex. The tensors C_α and C_β into which the curvature tensor splits are connected by the condition $\overline{C}_\alpha = C_\beta$ for the $CO(1, 3)$-structure, and by the conditions $\overline{C}_\alpha = C_\alpha$ and $\overline{C}_\beta = C_\beta$ for the $CO(4)$-structure.

The conformal structures for which the tensor C_α or C_β vanishes are called conformally semiflat. If both these tensors vanish, the structure is called conformally flat. The conditions which these tensors satisfy show that only the $CO(2, 2)$- and $CO(4)$-structures can be conformally semiflat, and that the $CO(1, 3)$-structure cannot be conformally semiflat without being conformally flat.

The curvature forms Θ_α and Θ_β of all four-dimensional conformal structures belong to eigensubspaces of the Hodge operator [H 41], and according to the terminology introduced in the paper [AHS 78], they are self-dual and anti-self-dual, respectively. By virtue of this, the β-semiflat conformal structures are self-dual while the α-semiflat conformal structures are anti-self-dual. Moreover, the $CO(1, 3)$-structure cannot be self-dual or anti-self-dual without being conformally flat.

Further, we consider two-dimensional completely isotropic submanifolds of four-dimen sional conformal structures. We prove that two-dimensional directions tangent to these submanifolds are principal, i.e. the parameters defining these directions satisfy one of two-fourth degree algebraic equations whose coefficients are the components of the tensors C_α and C_β. We establish a geometric meaning for semiflatness of four-dimensional conformal structures and find sufficient conditions for existence for integrable principal isotropic distributions on the bundles E_α and E_β.

Since space–time in general relativity is endowed with a conformal structure of signature $(1, 3)$, we were able to find a relationship between the Petrov classification of Einstein's spaces (see [Ch 83] or [PR 86]) with the structure of principal isotropic distributions defined on the $CO(1,3)$-structure and the integrability conditions for these distributions.

Note also that the real theory of four-dimensional Riemannian and pseudo-Riemannian metrics of different signatures and its applications to general relativity were considered in the recent paper [BGPPR 94].

The results of this paper were presented at the Conference on Differential Geometry and Its Applications (28 August–1 September, 1996; Brno, Czech Republic). Some of the results of the current paper were published earlier in [A 83].

1. Isotropic fiber bundles

1.1

Let M be a real differentiable manifold of dimension n, and g be a nondegenerate quadratic form of signature (p,q), $p+q=n$, given on M. A pair (M,g) is called a *Riemannian manifold* of signature (p,q), and the form g is called a *Riemannian metric* on M. For $q=0$, this metric is *proper Riemannian*, and for $0<q<n$, it is *pseudo-Riemannian*.

Two Riemannian metrics g and $\overline{g}$ are called *conformally equivalent* on the manifold M if $\overline{g}=\sigma g$ where $\sigma=\sigma(x)$ is a smooth function on M such that $\sigma(x)\neq 0$. If $\sigma(x)>0$, then the quadratic forms g and $\overline{g}$ have the same signature. If $\sigma(x)<0$, then the quadratic form $\overline{g}$ is of signature (q,p).

A *conformal structure* on a manifold M is the collection of all conformally equivalent Riemannian metrics given on M. It is denoted by $CO(p,q)$. It is easy to see that the conformal structures $CO(p,q)$ and $CO(q,p)$ are equivalent: $CO(p,q)\sim CO(q,p)$.

Let $T_x(M)$ be the tangent space to the manifold M at a point x, $\{e_i\}$, $i=1,\dots,n$, be a vectorial frame, and $\{\omega^i\}$ be the coframe dual to the frame $\{e_i\}$: $\omega^i(e_j)=\delta^i_j$. With respect to the frame $\{e_i\}$, the quadratic form g can be written as follows:

$$g=g_{ij}\omega^i\omega^j,\quad i,j=1,\dots,n, \tag{1.1}$$

where g_{ij} are the components of the nondegenerate symmetric metric tensor on M which is called the *metric tensor*.

Since the form (1.1) is of signature (p,q), then in a neighborhood of each point $x\in M$, this form can be reduced to a canonical form having p positive and q negative squares. The form (1.1) is invariant on the Riemannian manifold, and it is relatively invariant on the conformal structure.

The conformal structure $CO(p,q)$ is a G-structure whose structure group G is a subgroup of the general linear group $\mathbf{GL}(n)$ of transformations of the tangent space $T_x(M)$. The transformations of this subgroup transfer the equation

$$g_{ij}\omega^i\omega^j=0 \tag{1.2}$$

into itself. This subgroup is the direct product

$$G=\mathbf{SO}(p,q)\times\mathbf{H},\quad p+q=n, \tag{1.3}$$

where $\mathbf{SO}(p,q)$ is the special pseudoorthogonal group of signature (p,q) and $\mathbf{H} = \mathbf{R}^* \times \mathrm{Id}$ is the group of homotheties of the space $T_x(M)$ ($\mathbf{R}^*$ is the multiplicative group of reals).

Consider the complexified tangent space $\mathbf{C}T_x = T_x \otimes \mathbf{C}$. In this space, $T_x(M)$ is a real subspace, and Eq. (1.2) defines an *isotropic cone* C_x which can be real or imaginary. The group G acts in the space $\mathbf{C}T_x$ in such a way that if $\gamma \in G$ and $\xi \in T_x(M)$, then

$$\gamma T_x = T_x, \quad \gamma(\bar{\xi}) = \overline{\gamma(\xi)}, \quad \gamma(C_x) = C_x.$$

We will also consider the fibration $\mathbf{C}T(M)$ with the real base M and the complex fibers $\mathbf{C}T_x(M),\ x \in M$.

The assignment of a conformal structure $CO(p,q)$ on a real manifold M is equivalent to the assignment of a differentiable fibration of isotropic cone $C_x,\ C_x \subset \mathbf{C}T_x(M)$.

1.2

Consider conformal structures on a four-dimensional manifold M. There exist structures of three substantially different types on such a manifold: these are the structures $CO(4,0) = CO(4)$, $CO(1,3)$ and $CO(2,2)$.

For the $CO(4)$-structure, the quadratic form g can be reduced to the form

$$g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 + (\omega^4)^2. \tag{1.4}$$

Such a conformal structure is called a *proper conformal structure* or *a structure of elliptic type*.

For the $CO(1,3)$-structure, the quadratic form g can be reduced to the form

$$g = (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2 - (\omega^4)^2. \tag{1.5}$$

Such a conformal structure is called a *structure of Lorentzian type*.

Finally, for the $CO(2,2)$-structure, the quadratic form g can be reduced to the form

$$g = (\omega^1)^2 - (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2. \tag{1.6}$$

Such a conformal structure is called a *structure of ultrahyperbolic type*.

For the $CO(4)$-structure, the isotropic cones C_x are pure imaginary. For the $CO(1,3)$-structure, they are real cones of second order that have real rectilinear generators but do not have real planar generators. Finally, for the $CO(2,2)$-structure, these cones are real cones of second order that have real planar generators.

Let us find the equations of plane generators of isotropic cones of the $CO(2,2)$-structure. To this end, we will make the real transformation of coordinates in the space $T_x(M)$ under which

$$\begin{aligned} &\omega^1 + \omega^4 \to \sqrt{2}\,\omega^1, \qquad \omega^2 + \omega^3 \to \sqrt{2}\omega^2, \\ &\omega^1 - \omega^4 \to \sqrt{2}\,\omega^4, \qquad \omega^2 - \omega^3 \to \sqrt{2}\,\omega^3. \end{aligned} \tag{1.7}$$

Then the quadratic form g becomes

$$g = 2(\omega^1\omega^4 - \omega^2\omega^3), \tag{1.8}$$

where the forms ω^i are real. A vector $\xi \in T_x$ can be written as

$$\xi = \omega^i(\xi)e_i, \tag{1.9}$$

where e_i are real basis vectors of the space T_x connected by the conditions:

$$\begin{aligned} &(e_i, e_i) = 0, \qquad (e_1, e_2) = (e_1, e_3) = (e_4, e_2) = (e_4, e_3) = 0, \\ &(e_1, e_4) = -(e_2, e_3) = 1, \end{aligned} \tag{1.10}$$

where the parentheses denote the scalar product in the space T_x, which is defined by the quadratic form g. The first relations in (1.10) mean that the vectors e_i are isotropic. This is the reason that the frame formed by these vectors is called *isotropic*.

The equations $g = 0$ of the isotropic cone C_x can be written in two different forms:

$$\frac{\omega^1}{\omega^3} = \frac{\omega^2}{\omega^4} = -\lambda \quad \text{and} \quad \frac{\omega^1}{\omega^2} = \frac{\omega^3}{\omega^4} = -\mu,$$

and the latter equations can be written in the form:

$$\omega^1 + \lambda\omega^3 = 0, \qquad \omega^2 + \lambda\omega^4 = 0, \tag{1.11}$$

and

$$\omega^1 + \mu\omega^2 = 0, \qquad \omega^3 + \mu\omega^4 = 0. \tag{1.12}$$

Eqs. (1.11) and (1.12) determine two families of real two-dimensional plane generators—*isotropic planes*—of the isotropic cone C_x of $CO(2, 2)$-structure. The 2-planes of family (1.11) are called α*-planes*, and those of family (1.12) β*-planes*.

The parameters λ and μ in Eqs. (1.11) and (1.12) are nonhomogeneous projective coordinates on these families, These families are homeomorphic to the real projective lines $\mathbf{R}P_\alpha$ and $\mathbf{R}P_\beta$, respectively.

It follows that the group $\mathbf{SO}(2, 2)$ leaving the isotropic cone C_x invariant is decomposed into the direct product of two groups $\mathbf{SL}(2)$ of projective transformations of the real lines $\mathbf{R}P_\alpha$ and $\mathbf{R}P_\beta$. Thus, for the $CO(2, 2)$-structure, the structure group G is isomorphic to the direct product: $G \cong \mathbf{SL}(2) \times \mathbf{SL}(2) \times \mathbf{H}$, where $\mathbf{H}$ is the one-parameter group of homotheties of the space T_x. Moreover, two real groups $\mathbf{SL}(2)$ act independently on the families of α- and β-planes of the cone C_x.

The isotropic α- and β-planes of the $CO(2, 2)$-structure form two fiber bundles E_α and E_β with common base M and the α- and β-planes of the cones C_x as their fibers. Since these fibers are isomorphic to the projective lines $\mathbf{R}P_\alpha$ and $\mathbf{R}P_\beta$, respectively, we will write $E_\alpha = (M, \mathbf{R}P_\alpha)$ and $E_\beta = (M, \mathbf{R}P_\beta)$ and call these fiber bundles the *isotropic fiber bundles* of the $CO(2, 2)$-structure.

It follows from our consideration that on the pseudoconformal $CO(2, 2)$-structure the isotropic fiber bundles are real.

Consider further the $CO(1, 3)$-structure—the conformal structure of Lorentzian type whose fundamental quadratic form can be reduced to the form (1.5) by means of a real transformation of coordinates in the space T_x. This form can be reduced to the form (1.8) only by means of a complex transformation of coordinates. To this end, we must complexify the space T_x, i.e. to consider the space $\mathbf{C}T_x$. Now, by means of the complex transformation

$$\begin{aligned} \omega^1 + \omega^4 &\to \sqrt{2}\omega^1, & \omega^2 + \mathrm{i}\omega^3 &\to \sqrt{2}\omega^2, \\ \omega^1 - \omega^4 &\to \sqrt{2}\omega^4, & \omega^2 - \mathrm{i}\omega^3 &\to \sqrt{2}\omega^3, \end{aligned} \tag{1.13}$$

the quadratic form (1.5) can be reduced to the form (1.8). Moreover, the forms ω^1 and ω^4 are real, and the forms ω^2 and ω^3 are complex conjugate forms:

$$\overline{\omega}^1 = \omega^1, \quad \overline{\omega}^4 = \omega^4, \quad \overline{\omega}^2 = \omega^3. \tag{1.14}$$

Since as before the tangent vector $\xi \in T_x$ represented by (1.9) remains real, and on the $CO(1,3)$-structure the basis forms ω^i satisfy conditions (1.14), the basis vectors of the complexified space $\mathbf{C}T_x$ satisfy the relations

$$\overline{e}_1 = e_1, \quad \overline{e}_4 = e_4, \quad \overline{e}_2 = e_3. \tag{1.15}$$

Such a basis in the space $\mathbf{C}T_x$ is called the *tetrad of Newman–Penrose* (see [Ch 83]).

As before, the isotropic planes on the $CO(1,3)$-structure are determined by Eqs. (1.11) and (1.12) . However, the parameters λ and μ in these equations must be now considered as complex nonhomogeneous coordinates on the complex lines $\mathbf{C}P_\alpha$ and $\mathbf{C}P_\beta$. Moreover, the isotropic planes of the Lorentzian structure are two-dimensional complex generators of the cones C_x.

By (1.14), if in Eqs. (1.11) we replace all quantities by their conjugates, we obtain Eqs. (1.12), where $\mu = \overline{\lambda}$. Thus, the isotropic bundles $E_\alpha = (M, \mathbf{C}P_\alpha)$ and $E_\beta = (M, \mathbf{C}P_\beta)$ are complex conjugates: $\overline{E}_\beta = E_\alpha$.

On the cone C_x of the $CO(1,3)$-structure, there is a bijective correspondence between its α- and β-generators, and this correspondence is determined by the condition $\mu = \overline{\lambda}$. Moreover, two complex conjugate generators of the cone C_x intersect one another along its real rectilinear generator. The equation of this generator can be found from Eqs. (1.11) and (1.12) and condition $\mu = \overline{\lambda}$. Solving these equations, we find that

$$\omega^1 = \lambda\overline{\lambda}\omega^4, \quad \omega^2 = -\lambda\omega^4, \quad \omega^3 = -\overline{\lambda}\omega^4.$$

Hence the directional vector of the rectilinear generator can be written in the form

$$\xi = \lambda\overline{\lambda}e_1 - \lambda e_2 - \overline{\lambda}e_3 + e_4. \tag{1.16}$$

By condition (1.15), this vector is real. It depends on one complex parameter or two real parameters. Eqs. (1.16) can be considered as the equation of the director two-dimensional surface of the three-dimensional cone C_x in the real space $T_x(M)$.

Since the isotropic fiber bundles E_α and E_β of the Lorentzian structure $CO(1,3)$ are complex conjugates, its structural group G can be represented as follows:

$$G \cong \mathbf{SL}(2, \mathbf{C}) \times \mathbf{H} \cong \overline{\mathbf{SL}(2, \mathbf{C})} \times \mathbf{H},$$

where the groups $\mathbf{SL}(2, \mathbf{C})$ and $\overline{\mathbf{SL}(2, \mathbf{C})}$ act concordantly on the fiber bundles E_α and E_β. The group G depends on seven real parameters.

Finally, we consider the proper conformal structure $CO(4)$ whose fundamental quadratic form can be reduced to the form (1.4) by means of a real transformation of coordinates. Complexifying the tangent space and applying the transformation

$$\begin{aligned} &\omega^1 + \mathrm{i}\omega^4 \to \sqrt{2}\omega^1, \qquad \omega^2 + \mathrm{i}\omega^3 \to \sqrt{2}\omega^2, \\ &\omega^1 - \mathrm{i}\omega^4 \to \sqrt{2}\omega^4, \qquad \omega^2 - \mathrm{i}\omega^3 \to -\sqrt{2}\omega^3 \end{aligned} \tag{1.17}$$

in this space, we again reduce the form g to the form (1.8). However, now all the forms ω^i become complex forms related by the conditions

$$\omega^4 = \overline{\omega}^1, \qquad \omega^3 = -\overline{\omega}^2, \tag{1.18}$$

and the basis vectors of the space $\mathbf{C}T_x$ satisfy the conditions

$$e_4 = \bar{e}_1, \qquad e_3 = -\bar{e}_2. \tag{1.19}$$

As before, the parameters λ and μ in Eqs. (1.11) and (1.12) will be complex nonhomogeneous coordinates on the complex projective lines $\mathbf{C}P_\alpha$ and $\mathbf{C}P_\beta$, and the isotropic fiber bundles $E_\alpha = (M, \mathbf{C}P_\alpha)$ and $E_\beta = (M, \mathbf{C}P_\beta)$ are formed by complex two-dimensional generators of the isotropic cones C_x.

From Eqs. (1.18) it follows that on the proper conformal structure $CO(4)$, each of the systems of equations (1.11) and (1.12) remains invariant under passage to the complex conjugate values if the parameters λ and μ undergo the following transformation:

$$\lambda \to -1/\overline{\lambda}, \qquad \mu \to -1/\overline{\mu}. \tag{1.20}$$

In view of this, the isotropic fiber bundles E_α and E_β are self-conjugate: $\overline{E_\alpha} = E_\alpha$, $\overline{E_\beta} = E_\beta$. This implies that for the $CO(4)$-structure, the structure group G can be represented in the form: $G = G_\alpha \times G_\beta \times \mathbf{H}$ where G_α and G_β are the groups acting on the fiber bundles E_α and E_β, respectively. But by condition (1.3), $G_\alpha \times G_\beta = \mathbf{S0}(4)$, and the group $\mathbf{S0}(4)$ can be represented as the direct product $\mathbf{S0}(4) = \mathbf{SU}(2) \times \mathbf{SU}(2)$. As a result, we find that for the $CO(4)$-structure, $G = \mathbf{SU}(2) \times \mathbf{SU}(2) \times \mathbf{H}$, and two groups $\mathbf{SU}(2)$ act independently on the families of α- and β-planes of the isotropic cone C_x.

2. The structure equations and the curvature forms of the $CO(p,q)$-structure

2.1

First, we consider the structure equations of the conformal structure $CO(p,q)$ of general type given on a manifold M of dimension $n = p + q$. For $q = 0$ and $n = p$ in the orthonormal frame, these equations were developed by Cartan as far back as 1923 (see [C 23]). For arbitrary p and q in the general frame, they were developed in [AK 93]. Note also that these equations are given in the book [Ga 89].

On the manifold M, in addition to the 1-form $\omega = \{\omega^i\}$ with their values in the space $T_x(M)$ and defined in a first-order frame bundle, one can invariantly define a matrix

1-form $\theta = \{\theta^i_j\}$ and a scalar 1-form κ in a second-order frame bundle, and a covector form $\varphi = \{\varphi_i\}$ in the third-order frame bundle. Along with the tensor $g = \{g_{ij}\}$, these forms satisfy the following structure equations:

$$\nabla g = 0, \tag{2.1}$$

$$d\omega = \kappa \wedge \omega - \theta \wedge \omega, \tag{2.2}$$

$$d\kappa = -\varphi \wedge \omega, \tag{2.3}$$

$$d\theta = \varphi \wedge \omega - \theta \wedge \theta + (g\omega) \wedge (\varphi g^{-1}) + \Theta, \tag{2.4}$$

$$d\varphi = \varphi \wedge \kappa - \varphi \wedge \theta + \Phi. \tag{2.5}$$

In these formulas $\nabla g = \{\, dg_{ij} - g_{ik}\theta^k_j - g_{kj}\theta^k_i \}$, d is the operator of exterior differentiation, and $\wedge$ is the symbol of exterior multiplication. In addition, in all exterior products of 1-forms occurring in Eqs. (2.1)–(2.5) multiplication is performed row by column: for example, a detailed writing of Eq. (2.2) has the following form:

$$d\omega^i = \kappa \wedge \omega^i - \theta^i_j \wedge \omega^j.$$

In Eqs. (2.1)–(2.5), the forms Θ and Φ are the curvature forms of the conformal structure $CO(p,q)$.

Let us find the geometric meaning of the 1-forms θ, κ and φ occurring in the structure equations (2.1)–(2.5). To this end, we consider the restrictions of these equations to a fiber of a third-order frame bundle, i.e. we will assume that in these equations $\omega = 0$. Then since for $\omega = 0$, the curvature forms vanish, and Eqs. (2.3)–(2.5) take the form:

$$d\kappa = 0, \quad d\theta = -\theta \wedge \theta, \quad d\varphi = \varphi \wedge \kappa - \varphi \wedge \theta. \tag{2.6}$$

Eqs. (2.6) show that the form κ is an invariant form of the group **H** of homotheties that acts in the tangent space $T_x(M)$. The matrix form θ which besides Eq. (2.6) satisfies also Eq. (2.1) is an invariant form of the group $\mathbf{SO}(p,q)$ which, as the group **H**, maps the isotropic cone C_x into itself. Jointly, the forms θ and κ are invariant forms of the structural group $G \cong \mathbf{S0}(p,q) \times \mathbf{H}$ of the conformal structure $CO(p,q)$.

Next, we will clarify the geometric meaning of the covector form φ occurring in Eqs. (2.1)–(2.5). To this end, we consider compactification of the tangent space $T_x(M)$. This compactification can be constructed as follows. Since in the space T_x, the invariant cone C_x is fixed, this space is endowed with the structure of the pseudo-Euclidean space R^n_q of signature (p,q). In this space, we consider a manifold of hyperspheres defined in Cartesian coordinates $x = \{x^i\}$ by the equation

$$kg(x,x) + 2h(x) + 2l = 0, \tag{2.7}$$

where $g(x,x)$ is the quadratic form determined by the tensor g, $h = \{h_i\}$ is a covector, $h(x) = h_i x^i$, and k and l are scalars. The quantities k, h_i and l are homogeneous coordinates of the hypersphere (2.7). These numbers can be taken as coordinates of a point in the projective space P^{n+1}_x. The compactified tangent space $T_x(M)$, which we denote by $S_x(M)$, is identified with the submanifold of hyperspheres of zero radius. The latter submanifold is given in the space P_x^{n+1} by the equation

$$g(x, x) - 2x^0x^{n+1} = 0, \tag{2.8}$$

where $x^0 = k$, $x^{n+1} = l$ and $x = hg^{-1}$. Thus, after compactification, the tangent space $T_x(M)$ is enlarged by the point at infinity y with coordinates $(0, 0, \ldots, 0, 1)$ and by the isotropic cone C_y with the vertex at this point y whose equation is the same as the equation of the cone C_x, namely $g(x, x) = 0$.

Thus, the compactified tangent space $S_x(M)$ is a hyperquadric determined by Eq. (2.8) in the local projective space P_x^{n+1}. On this hyperquadric, the structure of a pseudoconformal space C_q^n of signature (p, q) arises, and the fundamental group of C_q^n is locally isomorphic to the group $\mathbf{SO}(n+2, q+1)$. The space C_q^n is conformally flat, and its structure equations coincide with Eqs. (2.1)–(2.5) provided that $\Theta = 0$ and $\Phi = 0$ in them. In the space C_q^n, we can consider a family of frames consisting of two points x and y and linearly independent hyperspheres a_i passing through these two points. If we denote by $\langle\ ,\ \rangle$ the scalar product with respect to the quadratic form occurring in the left-hand side of Eq. (2.8), then the elements of these frames satisfy the equations:

$$\begin{aligned} &\langle x, x\rangle = \langle y, y\rangle = \langle x, a_i\rangle = \langle y, a_i\rangle = 0, \\ &\langle a_i, a_j\rangle = g_{ij}, \quad \langle x, y\rangle = -1. \end{aligned} \tag{2.9}$$

The last two conditions are the normalization conditions which follow from Eq. (2.8). The equations of infinitesimal displacement of this frame have the form:

$$\begin{aligned} \mathrm{d}x &= \kappa x + a\omega, \\ \mathrm{d}a &= \varphi x + a\theta \qquad + (g\omega)y, \\ \mathrm{d}y &= \qquad a(\theta g^{-1}) \ -\kappa y, \end{aligned} \tag{2.10}$$

where we denote the system of hyperspheres a_i by $a = \{a_i\}$. As in Eqs. (2.1)–(2.5), in Eqs. (2.10) multiplication is carried out row by column. For $\Theta = 0$ and $\Phi = 0$, Eqs. (2.1)–(2.5) are the conditions of complete integrability of the system of differential equations (2.10). For more details on this see the paper [AK 93].

On the hyperquadric $S_x(M)$ which is compactification of tangent space $T_x(M)$, the point x is fixed. Thus, $\omega = 0$ on $S_x(M)$, and other forms occurring in Eqs. (2.10) are invariant forms of the subgroup of the group $\mathbf{SO}(n+2, q+1)$ that leaves invariant the point x. As Eqs. (2.10) show, for $\omega = 0$, the scalar form κ determines the homothety of the hyperquadric $S_x(M)$ with respect to the points x and y, the matrix form θ determines the rotation of this hyperquadric when the points x and y are fixed, and, finally, the form φ determines a displacement of the point y on the hyperquadric $S_x(M)$.

If we fix point x on the hyperquadric $S_x(M)$, we turn $S_x(M)$ into an n-dimensional pseudo-Euclidean space $R_q^n = S_x \setminus C_x$ of signature (p, q) (we recall that $n = p + q$). The forms κ, θ and φ are invariant forms of the group G' of motions of this space. We have

$$G' \cong (\mathbf{SO}(p, q) \times \mathbf{H}) \ltimes \mathbf{T}(n), \tag{2.11}$$

where $\mathbf{T}(n)$ is the subgroup of translations of the group G', and the symbol $\ltimes$ denotes the semidirect product.

Eqs. (2.1)–(2.5) show that the group G' is a differential prolongation of the original structure group G, and the structure itself is a differential-geometric structure of finite type 2 (see [K 72, p.9] or [S 64, Ch. VII, Section 3]).

Cartan called the equations of type (2.1)–(2.5) the equations of the *normal conformal connection* associated with the quadratic differential form (1.1) (see [C 23]).

2.2

Consider the curvature forms $\Theta = \{\Theta^i_j\}$ and $\Phi = \{\Phi_i\}$ of the conformal structure $CO(p,q)$. Their decompositions with respect to the basis forms are:

$$\Theta^i_j = C^i_{jkl}\omega^k \wedge \omega^l, \qquad \Phi_i = C_{ijk}\omega^j \wedge \omega^k. \tag{2.12}$$

The coefficients C^i_{jkl} form the *tensor of conformal curvature* of the $CO(p,q)$-structure in question. They allow one to construct the invariant tensor of conformal curvature

$$C_{ijkl} = g_{im}C^m_{jkl},$$

which also is called the *Weyl tensor*. It satisfies the same conditions

$$C_{ijkl} = -C_{jilk} = -C_{ijlk} = C_{klij}, \tag{2.13}$$

$$C_{ijkl} + C_{iklj} + C_{iljk} = 0, \tag{2.14}$$

as the curvature tensor of a Riemannian manifold, and the additional conditions

$$g^{il}C_{ijkl} = C^i_{jki} = 0, \tag{2.15}$$

i.e. the Weyl tensor is trace-free. Condition (2.15) distinguishes the tensor of conformal curvature from the Riemannian tensor. The tensor of conformal curvature as the Riemannian tensor is connected with a differential neighborhood of third order of the manifold M endowed with the $CO(p,q)$-structure.

If $n \geq 4$, then the quantities C_{ijk} occurring in Eqs. (2.12) do not form a tensor. They can be expressed linearly in terms of the covariant derivatives C_{ijklm} of the tensor of conformal curvature. It follows that if $n \geq 4$ and the tensor of conformal curvature vanishes, then also $C_{ijk} = 0$, and the $CO(p,q)$-structure is conformally flat.

3. The structure equations of the $CO(2,2)$-structure

3.1

First, we will find the structure equations of the group G of the $CO(2,2)$-structure. By means of a real transformation of coordinates, its fundamental form can be reduced to the form (1.8), and its fundamental tensor has the following matrix of components:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

This implies that Eqs. (2.1) take the following form:

$$\begin{aligned} &\theta_1^4 = \theta_2^3 = \theta_3^2 = \theta_4^1 = 0, \\ &\theta_2^4 = \theta_1^3, \quad \theta_4^2 = \theta_3^1, \quad \theta_3^4 = \theta_1^2, \quad \theta_4^3 = \theta_2^1, \\ &\theta_1^1 + \theta_4^4 = 0, \quad \theta_2^2 + \theta_3^3 = 0. \end{aligned} \tag{3.2}$$

Thus, the matrix form $\theta = \{\theta_j^i\}$ has only six independent components: θ_1^2, θ_2^1, θ_1^3, θ_3^1, θ_1^1 and θ_2^2 which together with the form κ are invariant forms of the structure group G of the $CO(2,2)$-structure.

In order to find the structure equations of the group G, we consider the first two groups of Eqs. (2.6) taking into account Eqs. (3.2). Then we obtain:

$$\mathrm{d}\kappa = 0, \tag{3.3}$$

$$\mathrm{d}\theta_1^3 = (\theta_1^1 + \theta_2^2) \wedge \theta_1^3, \quad \mathrm{d}(\theta_1^1 + \theta_2^2) = 2\theta_1^3 \wedge \theta_3^1, \quad \mathrm{d}\theta_3^1 = \theta_3^1 \wedge (\theta_1^1 + \theta_2^2), \tag{3.4}$$

$$\mathrm{d}\theta_1^2 = (\theta_1^1 - \theta_2^2) \wedge \theta_1^2, \quad \mathrm{d}(\theta_1^1 - \theta_2^2) = 2\theta_1^2 \wedge \theta_2^1, \quad \mathrm{d}\theta_2^1 = \theta_2^1 \wedge (\theta_1^1 - \theta_2^2), \tag{3.5}$$

Eq. (3.3) shows that for $\omega = 0$, the form κ is a total differential. This form is an invariant form of the one-parameter group **H** of homotheties that sends each plane generator of the cone C_x of the $CO(2,2)$-structure into itself. Eqs. (3.4) show that for $\omega = 0$, the forms θ_1^3, θ_3^1 and $\theta_1^1 + \theta_2^2$ are invariant forms of the three-parameter group G_α which is isomorphic to the group **SL**(2) that sends the family of α-planes of the cone C_x into itself and keeps its β-planes fixed. Similarly, it follows from Eqs. (3.5) that the forms θ_2^1, θ_1^2 and $\theta_1^1 - \theta_2^2$ are invariant forms of the three-parameter group G_β; the latter group is also isomorphic to the group **SL**(2), which sends the family of β-planes of the cone C_x into itself and keeps its α-planes fixed. This matches the fact mentioned in Section 1 that the structure group G of the conformal $CO(2,2)$-structure is isomorphic to the following direct product: $G \cong \mathbf{SL}(2) \times \mathbf{SL}(2) \times \mathbf{H}$. As follows from Eqs. (2.11), the prolonged group G' of the $CO(2,2)$-structure has the form

$$G' \cong (\mathbf{SL}(2) \times \mathbf{SL}(2) \times \mathbf{H}) \ltimes \mathbf{T}(4).$$

3.2

Let us find now the structure equations of the $CO(2,2)$-structure. By (3.2), on the $CO(2,2)$-structure, Eqs. (2.2) take the form:

$$\begin{aligned} d\omega^1 &= (\kappa - \theta_1^1) \wedge \omega^1 + \omega^2 \wedge \theta_2^1 + \omega^3 \wedge \theta_3^1, \\ d\omega^2 &= (\kappa - \theta_2^2) \wedge \omega^2 + \omega^1 \wedge \theta_1^2 + \omega^4 \wedge \theta_3^1, \\ d\omega^3 &= (\kappa + \theta_2^2) \wedge \omega^3 + \omega^1 \wedge \theta_1^3 + \omega^4 \wedge \theta_2^1, \\ d\omega^4 &= (\kappa + \theta_1^1) \wedge \omega^4 + \omega^2 \wedge \theta_1^3 + \omega^3 \wedge \theta_1^2, \end{aligned} \tag{3.6}$$

and Eqs. (2.4) can be reduced to the form:

$$\begin{aligned} d\theta_1^3 &= \varphi_1 \wedge \omega^3 + \varphi_2 \wedge \omega^4 + (\theta_1^1 + \theta_2^2) \wedge \theta_1^3 + \Theta_1^3, \\ d(\theta_1^1 + \theta_2^2) &= \varphi_1 \wedge \omega^1 + \varphi_2 \wedge \omega^2 - \varphi_3 \wedge \omega^3 - \varphi_4 \wedge \omega^4, \\ &\quad +2\theta_1^3 \wedge \theta_3^1 + \Theta_1^1 + \Theta_2^2, \\ d\theta_3^1 &= \varphi_3 \wedge \omega^1 + \varphi_4 \wedge \omega^2 + \theta_3^1 \wedge (\theta_1^1 + \theta_2^2) + \Theta_3^1, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} d\theta_1^2 &= \varphi_1 \wedge \omega^2 + \varphi_3 \wedge \omega^4 + (\theta_1^1 - \theta_2^2) \wedge \theta_1^2 + \Theta_1^2, \\ d(\theta_1^1 - \theta_2^2) &= \varphi_1 \wedge \omega^1 - \varphi_2 \wedge \omega^2 + \varphi_3 \wedge \omega^3 - \varphi_4 \wedge \omega^4 \\ &\quad +2\theta_1^2 \wedge \theta_2^1 + \Theta_1^1 - \Theta_2^2, \\ d\theta_2^1 &= \varphi_2 \wedge \omega^1 + \varphi_4 \wedge \omega^3 + \theta_2^1 \wedge (\theta_1^1 - \theta_2^2) + \Theta_2^1. \end{aligned} \tag{3.8}$$

Eqs. (3.7) show that *the exterior quadratic forms Θ_1^3, $\Theta_1^1 + \Theta_2^2$ and Θ_3^1 are the components of the curvature form Θ_α of the isotropic fiber bundle E_α, and the forms Θ_1^2, $\Theta_1^1 - \Theta_2^2$ and Θ_2^1 are the components of the curvature form Θ_β of the isotropic fiber bundle E_β.*

In order to find decompositions of these curvature forms with respect to the basis forms ω^i, we must find independent components of the tensor of conformal curvature C_{ijkl} of the conformal $CO(2, 2)$-structure.

Since the indices i, j, k, l take only four values $1, 2, 3, 4$, it follows from Eqs. (2.13) that this tensor has 21 essential components which satisfy 11 independent conditions arising from Eqs. (2.14) and (2.15):

$$\begin{aligned} &C_{1234} - C_{1324} + C_{1423} = 0, \\ &C_{1224} = C_{1334} = C_{1213} = C_{2434} = 0, \\ &C_{1314} - C_{1323} = C_{1424} - C_{2324} = 0, \\ &C_{1214} + C_{1223} = C_{1434} + C_{2334} = 0, \\ &C_{1414} = C_{2323} = C_{1234} + C_{1324}. \end{aligned} \tag{3.9}$$

Hence the tensor C_{ijkl} has 10 independent components in all. We denote them as follows:

$$\begin{aligned} &C_{1212} = a_0, \quad C_{1214} = a_1, \quad C_{1234} = a_2, \quad C_{1434} = a_3, \quad C_{3434} = a_4, \\ &C_{1313} = b_0, \quad C_{1314} = b_1, \quad C_{1324} = b_2, \quad C_{1424} = b_3, \quad C_{2424} = b_4. \end{aligned} \tag{3.10}$$

Computing the components of the curvature form Θ_α of the fiber bundle E_α and applying notations (3.10), we find that

$$\begin{aligned} \Theta_1^3 &= 2[a_0\omega^1\wedge\omega^2 + a_1(\omega^1\wedge\omega^4 - \omega^2\wedge\omega^3) + a_2\omega^3\wedge\omega^4], \\ \Theta_1^1 + \Theta_2^2 &= -4[a_1\omega^1\wedge\omega^2 + a_2(\omega^1\wedge\omega^4 - \omega^2\wedge\omega^3) + a_3\omega^3\wedge\omega^4], \\ \Theta_3^1 &= -2[a_2\omega^1\wedge\omega^2 + a_3(\omega^1\wedge\omega^4 - \omega^2\wedge\omega^3) + a_4\omega^3\wedge\omega^4]. \end{aligned} \tag{3.11}$$

Similarly, computing the components of the curvature form Θ_β of the fiber bundle E_β, we obtain

$$\begin{aligned} \Theta_1^2 &= 2[b_0\omega^1\wedge\omega^3 + b_1(\omega^1\wedge\omega^4 + \omega^2\wedge\omega^3) + b_2\omega^2\wedge\omega^4], \\ \Theta_1^1 - \Theta_2^2 &= -4[b_1\omega^1\wedge\omega^3 + b_2(\omega^1\wedge\omega^4 + \omega^2\wedge\omega^3) + b_3\omega^2\wedge\omega^4], \\ \Theta_2^1 &= -2[b_2\omega^1\wedge\omega^3 + b_3(\omega^1\wedge\omega^4 + \omega^2\wedge\omega^3) + b_4\omega^2\wedge\omega^4]. \end{aligned} \tag{3.12}$$

Eqs. (3.11) and (3.12) show that *the tensor of conformal curvature of the structure $CO(2,2)$ is decomposed into two subtensors $C_\alpha = \{a_u\}$ and $C_\beta = \{b_u\}$, $u = 0, 1, 2, 3, 4$, which are the curvature tensors of the isotropic fiber bundles E_α and E_β.*

For the $CO(2,2)$-structure both these tensors are real and independent of one another.

If one of the subtensors C_α and C_β vanishes, then the $CO(2,2)$-structure is called *conformally semiflat.* If both subtensors, C_α and C_β, vanish, then the tensor of conformal curvature of the $CO(2,2)$-structure also vanishes, and the $CO(2,2)$-structure itself becomes *conformally flat*, i.e. it is locally isomorphic to the structure of the pseudoconformal space C_2^4.

4. The curvature forms of the $CO(1,3)$- and $CO(4)$-structure

4.1

We now consider the pseudoconformal $CO(1,3)$-structure on a real four-dimensional manifold M. As in Section 1, we assume that the tangent spaces $T_x(M)$ to the manifold M are complexified, i.e. they are complemented to $\mathbf{C}T_x(M) = T_x \otimes \mathbf{C}$. But in the space $\mathbf{C}T_x(M)$, we consider only those linear transformations which preserve their real subspaces T_x, and also we considered the symmetry correspondence (the complex conjugacy) with respect to these subspaces.

By means of transformations of this kind, the fundamental form of the $CO(1,3)$-structure can be reduced to the form (1.8) in complex coordinates related by condition (1.14).

In these coordinates, on the $CO(1,3)$-structure, Eqs. (3.2), (3.6), (3.7) and (3.8) are still valid but some of differential forms occurring in these equations are complex.

After some computation, from exterior equations obtained by differentiation of relations (1.14) by means of structure equations (3.6), one can find the following relations for the 1-forms θ_j^i:

$$\overline{\theta}_1^1 = \theta_1^1, \quad \overline{\theta}_2^2 = -\theta_2^2, \quad \overline{\theta}_1^3 = \theta_1^2, \quad \overline{\theta}_3^1 = \theta_2^1. \tag{4.1}$$

Eqs. (4.1) show that the complex forms θ_j^i occurring in them are expressed in terms of precisely six linearly independent forms. This number is equal to the number of parameters on which the Lorentz group depends. These six forms are real invariant forms of the group $\mathbf{SO}(1,3)$. For $\omega = 0$, *the forms* θ^i_j *connected by relations* (4.1) *define a complex representation of this group.*

Since the form κ occurring in Eq. (2.2) is real (this form is an invariant of the real group $\mathbf{H}$ of homotheties), it follows from this equation and (1.14) that

$$\omega^1 \wedge (\varphi_1 - \overline{\varphi}_1) + \omega^2 \wedge (\varphi_2 - \overline{\varphi}_3) + \omega^3 \wedge (\varphi_3 - \overline{\varphi}_2) + \omega^4 \wedge (\varphi_4 - \overline{\varphi}_4) = 0. \quad (4.2)$$

Comparing Eqs. (4.3) with relations obtained from Eq. (4.1) by means of Eqs. (3.7) and (3.8), one can show that

$$\overline{\varphi}_1 = \varphi_1, \quad \overline{\varphi}_2 = \varphi_3, \quad \overline{\varphi}_4 = \varphi_4. \quad (4.3)$$

and

$$\overline{b}_u = a_u. \quad (4.4)$$

Eqs. (4.3) show that the forms φ_1 and φ_4 are real, and the forms φ_2 and φ_3 are complex conjugates. For $\omega = 0$, these forms define a complex representation of the group $\mathbf{T}(4)$ of parallel translations in the compactified tangent space $S_x(M)$.

Finally, Eqs. (4.4) show that *the curvature tensors* C_α *and* C_β *of the isotropic fiber bundles* E_α *and* E_β *of the* $CO(1,3)$*-structure are complex conjugates:* $\overline{C}_\beta = C_\alpha$. This matches the fact proved in Section 1 that the isotropic fiber bundles E_α and E_β of the $CO(1,3)$-structure are complex conjugates themselves: $\overline{E}_\beta = E_\alpha$. Similarly, the forms Θ_α and Θ_β of this structure are also complex conjugates: $\overline{\Theta}_\beta = \Theta_\alpha$.

It follows that if one of the tensors C_α or C_β of the $CO(1,3)$-structure vanishes, the other one vanishes too. This implies that *the* $CO(1,3)$*-structure cannot be conformally semiflat without being conformally flat.*

4.2

For the proper conformal structure $CO(4)$, the fundamental form again can be reduced to the form (1.8) in complex coordinates related by condition (1.18).

In these coordinates, on the $CO(4)$-structure, Eqs. (3.2), (3.6), (3.7) and (3.8) are again valid.

From exterior equations obtained by differentiation of relations (1.18) by means of structure equations (3.6), one can find the following relations for the 1-forms θ^i_j:

$$\theta^1_1 + \overline{\theta}^1_1 = 0, \quad \theta^2_2 + \overline{\theta}^2_2 = 0, \quad \theta^2_1 + \overline{\theta}^1_2 = 0, \quad \theta^3_1 + \overline{\theta}^1_3 = 0. \quad (4.5)$$

It follows again that the complex forms θ^i_j on the $CO(4)$-structure occurring in (4.5) are expressed in terms of precisely six linearly independent forms, and for $\omega = 0$, *these forms are invariant forms of the group* $\mathbf{SO}(4)$.

Further, by means of Eqs. (1.18), (2.2), (3.2), (3.6), (3.7), (3.8) and (4.5), one can prove that on the $CO(4)$-structure, the following relations hold:

$$\overline{\varphi}_4 = \varphi_1, \quad \overline{\varphi}_3 = -\varphi_2, \quad (4.6)$$

$$\begin{aligned} \bar{a}_0 &= a_4, \quad \bar{a}_1 = -a_3, \quad \bar{a}_2 = a_2, \\ \bar{b}_0 &= b_4, \quad \bar{b}_1 = -b_3, \quad \bar{b}_2 = b_2. \end{aligned} \tag{4.7}$$

Relations (4.6) show that there are two independent forms among the forms φ_i. For example, the forms φ_1 and φ_2 can be taken as independent forms. For $\omega = 0$, they determine a complex representation of the group $\mathbf{T}(4)$ of translations in the space $S_x(M)$.

From relations (4.7) it follows that the curvature tensors C_α and C_β of the isotropic fiber bundles E_α and E_β of the conformal structure $CO(4)$ are independent of one another but satisfy the conditions $\overline{C}_\alpha = C_\alpha$ and $\overline{C}_\beta = C_\beta$. Of course, this corresponds to the self-conjugacy of the isotropic fiber bundles E_α and E_β of the proper conformal structure $CO(4)$: $\overline{E}_\alpha = E_\alpha$ and $\overline{E}_\beta = E_\beta$ noted in Section 1.

Since the tensors C_α or C_β are independent of one another, *the $CO(4)$-structure can be α- or β- semiflat without being conformally flat.*

5. The hodge operator in four-dimensional conformal spaces

5.1

The Hodge tensor on an oriented Riemannian manifold (M, g) is constructed by means of its metric tensor g and the discriminant tensor e. If $\dim M = 4$ and e_i, $i = 1, 2, 3, 4$, compose a basis in the tangent space $T_x(M)$, then

$$g_{ij} = g(e_i, e_j), \qquad e_{ijkl} = V(e_i, e_j, e_k, e_l) = \sqrt{|g|}\, \epsilon_{ijkl}, \tag{5.1}$$

where $V(e_i, e_j, e_k, e_l)$ is the volume of the parallelepiped constructed on the vectors e_i, e_j, e_k and e_l, and

$$\epsilon_{ijkl} = \begin{cases} 1 & \text{if } i, j, k, l \text{ is an even permutation of the indices } 1, 2, 3, 4, \\ -1 & \text{if } i, j, k, l \text{ is an odd permutation of the indices } 1, 2, 3, 4, \\ 0 & \text{if at least one pair of these indices coincides.} \end{cases}$$

We assume that $V(e_1, e_2, e_3, e_4) > 0$.

The *Hodge tensor* is defined by the formula

$$h_{ij}{}^{kl} = e_{ijpq}\, g^{pk} g^{ql}, \tag{5.2}$$

where g^{ij} is the inverse tensor of the tensor g_{ij}.

It is easy to prove that the Hodge tensor is conformally invariant, i.e. it is not changed under a conformal transformation of the metric g: $g \to \sigma g$.

The Hodge tensor defines the linear operator on the six-dimensional space Λ^2 of exterior quadratic forms over the manifold M, $\dim M = 4$:

$$h : \Lambda^2 \to \Lambda^2.$$

For an exterior quadratic form $\alpha = \frac{1}{2}\alpha_{ij}\omega^i \wedge \omega^j \in \Lambda^2$, the form $\beta = h(\alpha) = \frac{1}{2}\beta_{ij}\omega^i \wedge \omega^j$ is defined as follows:

$$\beta_{ij} = h_{ij}{}^{kl}\alpha_{kl}.$$

This operator is called the *Hodge operator* and often is denoted by the symbol $*$: $\beta = *(\alpha)$.

We note the following properties of the Hodge operator:

1. *The Hodge operator is symmetric* since the tensor e_{ijkl} is symmetric with respect to the pair of bivector indices

$$e_{ijkl} = e_{klij}.$$

2. *The square of the Hodge operator satisfies the relation*

$$h^2 = \text{sign}\,(\det g) \cdot \text{Id},$$

where Id is the identity operator in Λ^2.

Note that for $\dim M = 4$, the sign of $\det g$ is preserved under transformations of the basis $\{e_i\}$. The sign also is preserved under multiplication of the quadratic form g by a factor $\sigma(x) \neq 0$. But from expressions (1.4)–(1.6) of the quadratic form g in a specialized frame one can see that for the $CO(4)$- and $CO(2,2)$-structure, $\det g > 0$, and for the $CO(1,3)$-structure, $\det g < 0$. By virtue of this

$$h^2(\sigma) = \text{Id} \tag{5.3}$$

for the $CO(4)$- and $CO(2,2)$-structure, and

$$h^2(\sigma) = -\text{Id} \tag{5.4}$$

for the $CO(1,3)$-structure.

3. It follows from Eq. (5.3) that for the $CO(4)$- and $CO(2,2)$-structures, the Hodge operator has two real triple eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ to which there correspond two real three-dimensional eigensubspaces in the space of 2-forms Λ^2, so that

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{5.5}$$

where $\Lambda^2_\pm$ are the eigensubspaces corresponding to these eigenvalues.

4. It follows from Eq. (5.4) that for the $CO(1,3)$-structure, the Hodge operator has two complex conjugate eigenvalues $\lambda_1 = \mathrm{i}$ and $\lambda_2 = -\mathrm{i}$. Formula (5.5) is still valid but the subspaces Λ^2_+ and Λ^2_- are complex conjugates.

For four-dimensional conformal structures of all three types, the eigensubspaces of the Hodge operator corresponding to the eigenvalue λ_1 are called *self-dual*, and the eigensubspaces of the Hodge operator corresponding to the eigenvalue λ_2 are called *anti-self-dual* (cf. [AHS 78]).

5.2

We will now compute the components of the Hodge tensor for the $CO(2,2)$-structure assuming that the tangent space $T_x(M)$ is referred to an isotropic frame $\{e_i\}$, $i = 1, 2, 3, 4$, in which its fundamental form (1.1) has the form (1.8). In such a frame, the matrix of coefficients of the form (1.1) and its inverse matrix can be written in the form (3.1).

The Hodge tensor is computed by formula (5.2) where now we have $e_{ijkl} = \epsilon_{ijkl}$. As a result, the matrix of its components has the form:

$$(h) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.6}$$

where the bivector indices (i, j) are ordered as follows:

$$(1,2), \quad (2,3), \quad (3,1), \quad (2,4), \quad (1,4), \quad (3,4).$$

As can be expected, the matrix of the operator h is symmetric.

From (5.6) it follows that the characteristic polynomial of the operator h can be written in the form

$$\det(h - \lambda \operatorname{Id}) = (1-\lambda)^3(1+\lambda)^3.$$

Thus, as can be predicted, the Hodge operator has two real triple eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

As usual, the eigenspaces of the operator h can be found from the equation

$$h(\alpha) = \lambda\alpha,$$

where $\alpha \in \Lambda^2$ and $\lambda = \pm 1$. It follows that to the eigenvalue $\lambda_1 = 1$, there corresponds the eigenspace determined by the equations

$$\alpha_{31} = 0, \quad \alpha_{14} + \alpha_{23} = 0, \quad \alpha_{24} = 0, \tag{5.7}$$

and to the eigenvalue $\lambda_2 = -1$, there corresponds the eigensubspace Λ^2_- defined by the equations

$$\alpha_{12} = 0, \quad \alpha_{14} - \alpha_{23} = 0, \quad \alpha_{34} = 0. \tag{5.8}$$

Thus, the forms

$$\omega^1 \wedge \omega^2, \quad \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3, \quad \omega^3 \wedge \omega^4 \tag{5.9}$$

form a basis of the space Λ^2_+, and the forms

$$\omega^1 \wedge \omega^3, \quad \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3, \quad \omega^2 \wedge \omega^4 \tag{5.10}$$

form a basis of the space Λ^2_-.

Consider further the proper conformal structure $CO(4)$. As we have noted in Section 1, its fundamental form (1.1) also can be reduced to the form of (1.8) in complex coordinates

ω^i connected by conditions (1.18). Thus, the basis forms of the eigensubspaces Λ^2_+ and Λ^2_- of the Hodge tensor are also complex forms satisfying the conditions

$$\overline{\omega^1 \wedge \omega^2} = \omega^3 \wedge \omega^4, \qquad \overline{\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3} = -(\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3) \tag{5.11}$$

and

$$\overline{\omega^1 \wedge \omega^3} = \omega^2 \wedge \omega^4, \qquad \overline{\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3} = -(\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3). \tag{5.12}$$

This means that *the self-dual and anti-self-dual eigensubspaces* Λ^2_+ *and* Λ^2_- *of the Hodge tensor of the* $CO(4)$*-structure satisfy the conditions* $\overline{\Lambda}^2_+ = \Lambda^2_+$ *and* $\overline{\Lambda}^2_- = \Lambda^2_-$, *i.e. they are self-conjugate.*

For the conformal structure $CO(1,3)$, the fundamental form (1.1) can be reduced to the form of (1.8) in complex coordinates satisfying conditions (1.18). Thus, the complex basis forms of the eigensubspaces Λ^2_+ and Λ^2_- of the Hodge tensor satisfy the conditions

$$\begin{aligned} &\overline{\omega^1 \wedge \omega^2} = \omega^1 \wedge \omega^3, \qquad \overline{\omega^3 \wedge \omega^4} = \omega^2 \wedge \omega^4, \\ &\overline{\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3} = \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3, \end{aligned} \tag{5.13}$$

i.e. they are complex conjugates. Thus *the eigensubspaces* Λ^2_+ *and* Λ^2_- *of the Hodge tensor of the* $CO(1,3)$*-structure are complex conjugates themselves:* $\overline{\Lambda}^2_+ = \Lambda^2_-$.

5.3

Now we will return to the study of the curvature forms of four-dimensional conformal structures. As we proved in Section 3, for the $CO(2,2)$-structure, these forms decompose into two groups Θ_α and Θ_β. The first group is formed by the curvature forms of the isotropic fiber bundle E_α, and the second one by the curvature forms of the isotropic fiber bundle E_β. The components of these forms can be calculated by formulas (3.11) and (3.12).

Comparing formulas (3.11) and (3.12) with the basis forms (5.10) and (5.11) of the eigensubspaces Λ^2_+ and Λ^2_- of the Hodge operator of the $CO(2,2)$-structure, we arrive at the following result: *The curvature form* Θ_α *of the isotropic fiber bundle* E_α *of the* $CO(2,2)$*-structure belongs to the eigensubspace* Λ^2_+ *of the Hodge operator, i.e. it is self-dual, and the curvature form* Θ_β *of the isotropic fiber bundle* E_β *belongs to the eigensubspace* Λ^2_-, *i.e. it is anti-self-dual.*

The curvature forms of the isotropic fiber bundles E_α and E_β of the conformal structures $CO(4)$ and $CO(1,3)$ enjoy similar properties since in the appropriate complex coordinates, they have the same form (3.11) and (3.12). Moreover, *to the relations* $\overline{\Lambda}^2_+ = \Lambda^2_+$ *and* $\overline{\Lambda}^2_- = \Lambda^2_-$ *between eigensubspaces of the Hodge operator of the* $CO(4)$*-structure, there correspond the relations* $\overline{E}_\alpha = E_\alpha$, $\overline{E}_\beta = E_\beta$ *and* $\overline{C}_\alpha = C_\alpha$, $\overline{C}_\beta = C_\beta$ *between its isotropic fiber bundles and their curvature tensors* (see Sections 1 and 4). Similarly, *to the relation* $\overline{\Lambda}^2_+ = \Lambda^2_-$ *between eigensubspaces of the Hodge operator of the* $CO(1,3)$*-structure, there correspond the relations* $\overline{E}_\alpha = E_\beta$ *and* $\overline{C}_\alpha = C_\beta$ *between its isotropic fiber bundles and their curvature tensors* (see again Sections 1 and 4).

Finally, we consider the semiflat four-dimensional conformal structures. If the curvature tensor C_β of the fiber bundle E_β vanishes, then its curvature form belongs to the self-dual eigensubspace Λ^2_+ of the Hodge operator. This is the reason that such structures are called *self-dual*. On the other hand, if the curvature tensor C_α of the fiber bundle E_α vanishes, then its curvature form belongs to the anti-self-dual eigensubspace Λ^2_- of the Hodge operator. This is the reason that such structures are called *anti-self-dual*.

Note that *the $CO(1,3)$-structure cannot be self-dual or anti-self-dual without being conformally flat*. This result immediately follows from the fact that for the $CO(1,3)$-structure, $\overline{C}_\beta = C_\alpha$.

6. Completely isotropic submanifolds of four-dimensional conformal structures

6.1

The tensor of conformal curvature of the conformal structure $CO(p,q)$ defines the quadratic form

$$C(p) = C_{ijkl}\, p^{ij} p^{kl}$$

in the bivector space V_x^2 associated with the tangent space $T_x(M)$ to a manifold endowed with $CO(p,q)$-structure. We will call the quadratic form $C(p)$ the *relative conformal curvature* of the bivector p.

Let us compute the quadratic form $C(p)$ for the $CO(2,2)$-structure. Taking into account that the essential components of the conformal curvature are expressed by formula (3.10), we find the following expression for $C(p)$:

$$\begin{aligned}\tfrac{1}{4}C(p) = {}& a_0(p^{12})^2 + 2a_1 p^{12}(p^{14} - p^{23}) + a_2[2p^{12}p^{34} + (p^{14} - p^{23})^2] \\ & + 2a_3 p^{34}(p^{14} - p^{23}) + a_4(p^{34})^2 + b_0(p^{13})^2 + 2b_1 p^{13}(p^{14} + p^{23}) \\ & + b_2[-2p^{13}p^{42} + (p^{14} + p^{23})^2] - 2b_3 p^{42}(p^{14} + p^{23}) + b_4(p^{42})^2. \quad (6.1)\end{aligned}$$

Next, we find the values of the form $C(p)$ for isotropic bivectors belonging to α- and β-plane generators of the isotropic cone C_x of the $CO(2,2)$-structure. α-generators of the cone C_x are determined by the system of equations (1.11). Thus a bivector belonging to the α-plane $\alpha(\lambda)$ is determined by the vectors

$$\xi_\lambda = e_3 - \lambda e_1, \qquad \eta_\lambda = e_4 - \lambda e_2.$$

Hence the coordinates of the bivector $p_\lambda = \xi_\lambda \wedge \eta_\lambda$ are the minors of the matrix

$$\begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \end{pmatrix},$$

i.e. they are

$$p^{12} = \lambda^2, \quad p^{13} = 0, \quad p^{14} = -\lambda, \quad p^{23} = \lambda, \quad p^{34} = 1, \quad p^{42} = 0.$$

Substituting these expressions into Eqs. (6.1), we find that

$$\tfrac{1}{4}C(p_\lambda) = a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 := C_\alpha(\lambda). \tag{6.2}$$

In exactly the same way, for the bivector $p_\mu = \xi_\mu \wedge \eta_\mu$ belonging to a β-plane determined by the system of equations (1.12) we obtain

$$\tfrac{1}{4}C(p_\mu) = b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 := C_\beta(\mu). \tag{6.3}$$

Expressions (6.2) and (6.3) show that *the relative conformal curvature of the isotropic α- and β-bivectors is expressed by the polynomials of fourth degree whose coefficients are the components of the subtensors C_α and C_β of the tensor of conformal curvature of the $CO(2,2)$-structure.*

The isotropic bivectors for which relative conformal curvature vanishes, i.e. for which $C_\alpha(\lambda) = 0$ or $C_\beta(\mu) = 0$, are called the *principal isotropic bivectors.*

Since polynomials (6.2) and (6.3) are of fourth degree, it follows that in general, the isotropic cone C_x carries four principal α-planes and the same quantity of principal β-planes if we count each of these planes as many times as its multiplicity.

If a $CO(2,2)$-structure is α-semiintegrable, then Eq. (6.2) becomes an identity, and all α-planes of the cones C_x are principal planes. Similarly, for a β-semiflat $CO(2,2)$-structure, all β-planes of the cones C_x are principal planes. Finally, for a conformally flat $CO(2,2)$-structure, all its plane generators of the cones C_x are principal planes.

6.2

A two-dimensional submanifold V of the manifold M endowed with a pseudoconformal $CO(2,2)$-structure is called *completely isotropic* if all its tangent subspaces $T_x(V)$ are isotropic planes. If all of them belong to the isotropic fiber bundle E_α, then the isotropic submanifold is denoted by V_α, and if all of them belong to the isotropic fiber bundle E_β, then the isotropic submanifold is denoted by V_β.

The submanifold V_α is determined on M by the system of equations (1.11). On this submanifold, the 1-forms ω^3 and ω^4 are independent. Taking the exterior derivatives of Eqs. (1.11), we obtain the system of equations

$$\theta_\lambda \wedge \omega^3 = 0, \quad \theta_\lambda \wedge \omega^4 = 0, \tag{6.4}$$

where

$$\theta_\lambda := \mathrm{d}\lambda + \lambda(\theta_1^1 + \theta_2^2) - \theta_3^1 + \lambda^2\theta_1^3. \tag{6.5}$$

From (6.4) it follows that on the submanifolds V_α

$$\theta_\lambda = 0. \tag{6.6}$$

By taking the exterior derivative of this equation, excluding $\mathrm{d}\lambda$, and setting the coefficient of the product $\omega^3 \wedge \omega^4$ equal to zero, we obtain the equation

$$a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 = 0, \tag{6.7}$$

whose left-hand side coincides with the polynomial $C_\alpha(\lambda)$.

In exactly the same way, taking the exterior derivatives of Eqs. (1.12) which determine the submanifolds V_β, we arrive at another Pfaffian equation:

$$\theta_\mu := \mathrm{d}\mu + \mu(\theta_1^1 - \theta_2^2) - \theta_2^1 + \mu^2\theta_1^2 = 0. \tag{6.8}$$

From this equation, just as above, we obtain the following algebraic equation:

$$b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 = 0, \tag{6.9}$$

whose left-hand side coincides with the polynomial $C_\beta(\mu)$.

From Eqs. (6.7) and (6.9) it follows that *if on a manifold M endowed with a $CO(2,2)$-structure, there are completely isotropic submanifolds V_α or V_β, then all tangent subspaces of these submanifolds are principal α- or β-planes, respectively.*

Pfaffian equations (1.11) and (6.6) determine a distribution $\Delta(\alpha)$ of two-dimensional elements on the five-dimensional fiber bundle E_α. If this distribution is involutive, then it has a three-parameter family of integral surfaces, which are projected onto the manifold M as completely isotropic two-dimensional submanifolds V_α. But the condition of involutivity of the distribution $\Delta(\alpha)$ is complete integrability of the system of equations (1.11) and (6.6), and this condition reduces to the identically satisfying Eq. (6.7). The latter condition is equivalent to the vanishing of the subtensor C_α of the fiber bundle E_α, i.e. to the α-semiflatness of the $CO(2,2)$-structure. A similar conclusion is valid for a distribution $\Delta(\beta)$ determined by the system of equations (1.12) and (6.8).

This implies the following result:

The conformal structure $CO(2,2)$ is α-semiflat if and only if it carries a three-parameter family of completely isotropic submanifolds V_α, and it is β-semiflat if and only if it carries a three-parameter family of completely isotropic submanifolds V_β. The conformal structure $CO(2,2)$ is conformally flat if and only if it carries two three-parameter families of completely isotropic two-dimensional submanifolds.

The conformally flat $CO(2,2)$-structure is locally equivalent to the structure of the four-dimensional pseudoconformal space C_2^4, and in turn, the latter space can be mapped onto a hyperquadric Q_2^4 of a projective space P^5. This mapping is called the Darboux mapping. Under this mapping, *to completely isotropic submanifolds of the conformally flat $CO(2,2)$-structure, there correspond two-dimensional plane generators of the hyperquadric Q_2^4.*

6.3

Let us assume now that Eq. (6.7) does not vanish identically. Then it has four roots λ_p, $p = 1,2,3,4$, if we count each of these roots as many times as its multiplicity. Each of these roots determines a cross-section $s_p(\alpha) : M \to E_\alpha$ which is a *principal isotropic distribution* $\Delta_p(\alpha)$ of the fiber bundle E_α.

The principal distributions $\Delta_p(\alpha)$ are, generally speaking, not integrable, because the root λ_p of Eq. (6.7) may not satisfy Eq. (6.6). However, if this root satisfies Eq. (6.6), then the distribution $\Delta_p(\alpha)$ is integrable and determines an isotropic foliation $F_p(\alpha)$ on the manifold M.

In order to find an integrability condition of the principal distribution $\Delta_p(\alpha)$, we will suppose that $\lambda = \lambda_p$ in (6.7), and differentiate the resulting identity. Replacing $d\lambda$ with the help of (6.6) and making use of (6.6), we obtain

$$\begin{aligned} &dC_\alpha(\lambda) + 2C_\alpha(\lambda)(\kappa + \theta_1^1 + \theta_2^2) + 4\lambda C_\alpha(\lambda)\theta_1^3 \\ &\quad = (a_{0i}\lambda^4 - 4a_{1i}\lambda^3 + 6a_{2i}\lambda^2 - 4a_{3i}\lambda + a_{4i})\omega^i = 0, \quad i = 1, 2, 3, 4, \end{aligned}$$

where a_{ui} and further b_{ui}, $u = 0, 1, 2, 3, 4$, are the covariant derivatives of the components of the tensors C_α and C_β, respectively. Since the forms ω^i are linearly independent, *a root λ_p of Eq.* (6.7) *determines an integrable distribution $\Delta_p(\alpha)$ if and only if it satisfies the equations*

$$a_{0i}\lambda^4 - 4a_{1i}\lambda^3 + 6a_{2i}\lambda^2 - 4a_{3i}\lambda + a_{4i} = 0. \tag{6.10}$$

In exactly the same way, *the principal isotropic distribution $\Delta_p(\beta)$, determined by a root μ_p of Eq.* (6.9), *is integrable if and only if this root satisfies the equations*

$$b_{0i}\mu^4 - 4b_{1i}\mu^3 + 6b_{2i}\mu^2 - 4b_{3i}\mu + b_{4i} = 0. \tag{6.11}$$

We deduce some consequences of this result.

1. A pseudoconformal $CO(2, 2)$-structure is called *α-semirecurrent* if its tensor C_α satisfies the condition

$$a_{ui} = k_i a_u \tag{6.12}$$

and *β-semirecurrent* if its tensor C_β satisfies the condition

$$b_{ui} = l_i b_u, \tag{6.13}$$

where $u = 0, 1, 2, 3, 4$ and $i = 1, 2, 3, 4$. A pseudoconformal $CO(2, 2)$-structure is called *recurrent* if

$$\nabla C = \sigma C, \tag{6.14}$$

where C is the tensor of conformal curvature, and σ is a 1-form (cf. [AM 67]).

From Eqs. (6.10) and (6.11) it follows that *the $CO(2, 2)$-structure is α-semirecurrent if and only if all four of its principal distributions $\Delta_p(\alpha)$ are integrable; this structure is β-semirecurrent if and only if all four of its principal distributions $\Delta_p(\beta)$ are integrable; and finally, this structure is recurrent if and only if all eight of its principal distributions are integrable.*

2. *Every multiple root of Eq.* (6.7) *or* (6.9) *determines a principal isotropic foliation on the manifold M.*

For example, let λ be a multiple root of Eq. (6.7). Using an admissible transformation of the adapted frame, we can set this root equal to zero, $\lambda = 0$. Then from (6.7) we conclude that $a_3 = a_4 = 0$. In view of this, we obtain $a_{4i} = 0$. But then the root $\lambda = 0$ satisfies Eqs. (6.10), and the distribution defined by it is integrable.

6.4

Consider now a $CO(1,3)$-structure. As we have proved earlier (see Eqs. (4.4)), for such a structure the coefficients of the polynomials $C_\alpha(\lambda)$ and $C_\beta(\mu)$ are complex conjugate. By virtue of this, the roots of these polynomials are also complex conjugates. Thus, the principal two-dimensional directions on the isotropic bundles E_α and E_β (which also satisfy the condition $\overline{E}_\beta = E_\alpha$) are also complex conjugate. Moreover, two complex conjugate two-dimensional principal directions of the bundles E_α and E_β determined by the roots λ_p and $\mu_p = \overline{\lambda}_p$ of Eqs. (6.7) and (6.9) intersect one another along a real generator of the cone C_x. This generator has the same direction as the vector ξ_p, $p = 1, 2, 3, 4$, defined by formula (1.16) for $\lambda = \lambda_p$. Thus, the isotropic cone C_x of the $CO(1,3)$-structure carries four real principal isotropic directions.

Let us prove that *the integral curves of each of four families of real principal directions on a manifold M with a $CO(1,3)$-structure are isotropic geodesics of the manifold M.*

Note first that in general, the geodesics of conformally equivalent Riemannian metrics generating a conformal structure on the manifold M are not conformally invariant. However, it is possible to prove that *the isotropic geodesics on M enjoy this property.*

The equations of geodesics on a Riemannian manifold M can be written in the form

$$d\xi^i + \xi^i\theta^i_j = \kappa\xi^i, \tag{6.15}$$

where ξ^i are coordinates of vectors tangent to the geodesics. For four-dimensional conformal structures, in the isotropic frame bundle the forms θ^i_j satisfy relations (3.2), and by (1.16), the coordinates of isotropic vectors on the $CO(1,3)$-structure have the form:

$$\xi^1 = \lambda\overline{\lambda}, \quad \xi^2 = -\lambda, \quad \xi^3 = -\overline{\lambda}, \quad \xi^4 = 1, \tag{6.16}$$

where λ is a complex parameter on the cone C_x. By virtue of (6.16), Eqs. (6.15) of isotropic geodesics on the $CO(1,3)$-structure can be written as follows:

$$\begin{aligned} &d(\lambda\overline{\lambda}) - \lambda\theta^1_2 - \overline{\lambda}\theta^1_3 = \lambda\overline{\lambda}(\kappa - \theta^1_1),\\ &-d\lambda + \lambda\overline{\lambda}\theta^2_1 + \theta^1_3 = -\lambda(\kappa - \theta^2_2),\\ &-d\overline{\lambda} + \lambda\overline{\lambda}\theta^3_1 + \theta^1_2 = -\overline{\lambda}(\kappa + \theta^2_2),\\ &-\lambda\theta^3_1 - \overline{\lambda}\theta^2_1 = \kappa + \theta^1_1. \end{aligned} \tag{6.17}$$

By relations (4.1), which the forms θ^i_j of the $CO(1,3)$-structure satisfy, only two of Eqs. (6.17), for example, the second and the fourth, are independent. Excluding the 1-form κ from the second equation by means of the fourth equations, we find that

$$d\lambda + \lambda(\theta^1_1 + \theta^2_2) - \theta^1_3 + \lambda^2\theta^3_1 = 0. \tag{6.18}$$

But this equation precisely coincides with Eq. (6.6) which the complex parameters λ_p determining the principal directions on the isotropic cones C_x satisfy. This proves the result formulated above.

Note also that integral curves of the principal isotropic directions of the $CO(1,3)$-structure form isotropic geodesic congruences on the manifold M. In general, the manifold M carries four such congruences.

As we will see further, the real principal directions on the isotropic cones C_x of the $CO(1,3)$-structure play an important role in the Petrov classification (see [Ch 83] or [PR 86]) of Riemannian metrics in general relativity.

6.5

In conclusion we consider a $CO(4)$-structure. By (4.7), Eq. (6.7) takes the form

$$a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 + 4\bar{a}_1\lambda + \bar{a}_0 = 0, \tag{6.19}$$

where a_2 is a real number. If we take the complex conjugate values of all terms of (6.19), we obtain

$$\bar{a}_0\bar{\lambda}^4 - 4\bar{a}_1\bar{\lambda}^3 + 6a_2\bar{\lambda}^2 + 4a_1\bar{\lambda} + a_0 = 0.$$

Comparing this equation with Eq. (6.19), we see that if λ_1 is a root of Eq. (6.19), then the number $\lambda_2 = -1/\bar{\lambda}_1$ is also its root. It follows that the roots λ_1 and λ_2 cannot coincide. Furthermore, if $\lambda_1 = \lambda_3$, then $\lambda_2 = \lambda_4$. Thus, we have proved the following result: *Eq. (6.19) has either four distinct roots or two pairs of double roots; in the latter case the isotropic fiber bundle E_α carries two double principal distributions.* However, since these distributions are complex, they do not define foliations on the real manifold M.

By (4.7), for the isotropic fiber bundle E_β, the equation $C_\beta(\mu) = 0$ can be written in the form

$$b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 + 4\bar{b}_1\mu + \bar{b}_0 = 0, \tag{6.20}$$

where b_2 is a real number. By means of Eq. (6.20), we can prove the results on the principal distributions of the isotropic fiber bundle E_β similar to those we proved above for the principal distributions of the isotropic fiber bundle E_α.

6.6

For the $CO(1,3)$-structure, Eqs. (6.7) and (6.9), which by (4.4) are complex conjugates of one another, are connected with the classification of A.Z. Petrov of Einstein spaces.

We remind the reader that the *Einstein space* is a four-dimensional pseudo-Riemannian manifold of signature (1, 3) whose curvature tensor R^i_{jkl} satisfies the condition

$$R_{jk} - \frac{1}{2}g_{jk}R = -\frac{8\pi G}{c^2}T_{ij}, \tag{6.21}$$

where $R_{jk} = R^i_{jki}$ is the Ricci tensor, $R = g^{jk}R_{jk}$ is the scalar curvature of the Riemannian manifold, T_{ij} is the energy-momentum tensor, G is the gravitational constant, and c is the speed of light. Eq. (6.21) is called the *Einstein equation.*

In empty space, i.e. in a region of space–time in which $T_{ij} = 0$, the Einstein equation can be reduced to the form

$$R_{ij} = 0.$$

Thus, the curvature tensor of this space coincides with the Weyl tensor: $R^i_{jkl} = C^i_{jkl}$. This follows from the expression of the tensor C^i_{jkl} in terms of R^i_{jkl}, R_{jk} and R (see, for example, the book [Ch 83, Ch. 1, Section 7] or the paper [AK 93]).

The classification of Einstein spaces is connected with the structure of its tensor of conformal curvature. Hence this classification is of a conformal nature. This classification was first constructed by Petrov in the paper [Pe 54] (see also [Pi 57]). This classification is presented in detail in many books in general relativity (see, for example, the books [Ch 83, Ch. 1, Section 9; PR 86, Ch. 8]. However, to our knowledge, the relationship of this classification with integrability of principal isotropic distributions has not been considered before now.

To give a geometric characterization of Einstein spaces of different types, we will also apply isotropic geodesic on the manifolds endowed with a $CO(1, 3)$-structure which we considered in Section 6.4.

Since for the $CO(1, 3)$-structure, Eqs. (6.7) and (6.9) are complex conjugates, for classification of Einstein spaces it is sufficient to consider only one of these equations, for example, the first one. As a result, the Petrov classification can be presented in the form of the following table:

Petrov's type	Roots of the equation $C_\alpha(\lambda) = 0$	Characterization of principal distributions	Characterization of isotropic geodesic congruences
I	$\lambda_p \neq \lambda_q,\ p \neq q,\ p, q = 1, 2, 3, 4$	4 different of general type	4 simple
II	$\lambda_1 = \lambda_2 \neq \lambda_3, \lambda_4;\ \lambda_3 \neq \lambda_4$	1 double and 2 of general type	1 double and 2 simple
D	$\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$	2 double	2 double
III	$\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$	1 triple and 1 of general type	1 triple and 1 simple
N	$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$	1 quadruple	1 quadruple

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